ISRAEL JOURNAL OF MATHEMATICS 114 (1999), 1–27

FRACTIONAL INTEGRALS AND WAVELET TRANSFORMS ASSOCIATED WITH BLASCHKE–LEVY REPRESENTATIONS ON THE SPHERE

BY

BORIS RUBIN*

Department of Mathematics, The Hebrew University
Jerusalem 91904, Israel
email: boris@math.huji.ac.il

ABSTRACT

A family of the spherical fractional integrals $T^\alpha f = \gamma_{n,\alpha} \int_{\Sigma_n} |xy|^{\alpha-1} f(y)dy$ on the unit sphere Σ_n in $\mathbb{R}^{n+1}$ is investigated. This family includes the spherical Radon transform ($\alpha = 0$) and the Blaschke–Levy representation ($\alpha > 1$). Explicit inversion formulas and a characterization of $T^\alpha f$ are obtained for f belonging to the spaces C^∞, C, L^p and for the case when f is replaced by a finite Borel measure. All admissible $n \geq 2$, $\alpha \in \mathbb{C}$, and p are considered. As a tool we use spherical wavelet transforms associated with T^α. Wavelet type representations are obtained for $T^\alpha f, f \in L^p$, in the case $\operatorname{Re}\alpha \leq 0$, provided that T^α is a linear bounded operator in L^p.

0. Introduction

Our investigation is motivated by the following problems: (1) How to define wavelet transforms on the sphere. (2) How to invert integral operators

$$(0.1) \quad (B_q f)(x) = \int_{\Sigma_n} |xy|^q f(y)dy, \quad (Rf)(x) = \frac{1}{|\Sigma_{n-1}|} \int_{\{y\in\Sigma_n : xy=0\}} f(y)d\sigma(y)$$

and to characterize their ranges, e.g., for $f \in L^p(\Sigma_n)$ or $f \in C(\Sigma_n)$. Instead of f, a finite Borel measure on Σ_n can be considered. Different approaches to (1) can be found in the papers by W. Freeden and U. Windheuser, J. Goettelmann,

* Partially supported by the Edmund Landau Center for Research in Mathematical Analysis, sponsored by the Minerva Foundation (Germany).
Received February 3, 1997 and in revised form February 15, 1998

M. Holschneider, P. Schröder and W. Sweldens, B. Torresani; see also [11]. The integral $F = B_q f$, where $q > 0$ is not an even integer, is called the Blaschke–Levy representation of F ([1, 8]). Among the authors, who studied this representation, are A. D. Aleksandrov, R. J. Gardner, P. R. Goodey, H. Groemer, A. Koldobsky, R. Schneider, W. Weil (for more information see [3, 6]). The inversion formula for $F = B_q f$ was obtained by Koldobsky [6] for $q > 0$ excluding the cases (a) q even and (b) q odd, n even. His method employs the Fourier transform on $\mathbb{R}^{n+1}$ (see also the earlier paper by Semyanistyi [16]). The second integral in (0.1) is known as the spherical Radon transform [5, 12].

We study the more general analytic family

$$(0.2)\qquad (T^\alpha f)(x) = \frac{\Gamma((1-\alpha)/2)}{2\pi^{n/2}\Gamma(\alpha/2)} \int\limits_{\Sigma_n} |xy|^{\alpha-1} f(y)dy, \quad \alpha \in \mathbb{C}, \quad \alpha \neq 1, 3, 5, \ldots,$$

arising in evaluation of the Fourier transform of homogeneous functions [15] (if $\mathrm{Re}\,\alpha \leq 0$, then (0.2) is understood in the sense of analytic continuation; for $\alpha = 1, 3, \ldots$, see [13]). By Corollary 2.6 (see below), T^α is bounded in $L^p(\Sigma_n)$, $1 < p < \infty$, if and only if $\mathrm{Re}\,\alpha \geq (1-n)/2 + |1/p - 1/2|(n-1)$. Let us explain the connection between the problems (1) and (2). Since $T^\alpha f \equiv 0$ for f odd, in the following f is assumed to be even. By using the formula

$$(0.3)\qquad \int\limits_{\Sigma_n} a(xy) f(y) dy = \sigma_{n-1} \int\limits_{-1}^{1} a(\tau)(M_\tau f)(x)(1-\tau^2)^{n/2-1} d\tau,$$

$$(0.4)\qquad (M_\tau f)(x) = \frac{(1-\tau^2)^{(1-n)/2}}{\sigma_{n-1}} \int\limits_{xy=\tau} f(y) d\sigma(y),$$

$$\tau \in (-1, 1), \quad \sigma_{n-1} = |\Sigma_{n-1}|,$$

we write $T^\alpha f$ in the "one-dimensional" form

$$T^\alpha f = \tilde{\gamma}_{n,\alpha} \int\limits_0^1 \tau^{\alpha/2-1}(1-\tau)^{n/2-1} M_{\sqrt{\tau}} f \, d\tau, \quad \tilde{\gamma}_{n,\alpha} = \frac{\sigma_{n-1}\Gamma((1-\alpha)/2)}{2\pi^{n/2}\Gamma(\alpha/2)},$$

$$0 < \mathrm{Re}\,\alpha < 1.$$

Put $\tau = ts$, then multiply both sides by $s^{-\alpha/2}$, and integrate with respect to an arbitrary sufficiently nice measure μ such that $\delta_{\alpha,\mu} \equiv \int_0^\infty s^{-\alpha/2} d\mu(s) \neq 0$. After changing the order of integration we get

$$(0.5)\qquad T^\alpha f = \frac{1}{c} \int\limits_0^\infty (W_\mu f)(x,t) \frac{dt}{t^{1-\alpha/2}}, \quad c = 2\pi^{n/2}\delta_{\alpha,\mu}\Gamma(\alpha/2)/\Gamma((1-\alpha)/2),$$

$$(W_\mu f)(x,t) = \sigma_{n-1} \int_0^{1/t} (1-ts)^{n/2-1} M_{\sqrt{ts}} f \, d\mu(s). \tag{0.6}$$

As we shall see below, (0.5) can be extended analytically to Re $\alpha \leq 0$ provided that μ enjoys some cancellation. The integral (0.6) will be called *the continuous wavelet transform of f generated by the wavelet measure μ and associated with the operator family $\{T^\alpha\}$.*

By choosing μ in a suitable way one can write (0.6) in different forms. For example, if μ is absolutely continuous, i.e. $d\mu(s) = w(s)ds$, then (0.3) yields $W_\mu f = Wf$, where

$$(Wf)(x,t) = \frac{1}{t} \int_{\Sigma_n} |xy| w(|xy|^2/t) f(y) dy. \tag{0.7}$$

By putting $u(s) = sw(s^2)$, $t = \tau^2$, $\tau > 0$, we get $(Wf)(x,\tau^2) = (W_u f)(x,\tau)$ where

$$(W_u f)(x,\tau) = \frac{1}{\tau} \int_{\Sigma_n} u(|xy|/\tau) f(y) dy. \tag{0.8}$$

The wavelet transform (0.8) was introduced in [12]. If

$$\mu = \sum_{k=0}^{\ell} (-1)^k \binom{\ell}{k} \delta_k, \quad \ell \in \mathbb{N}, \tag{0.9}$$

where $\delta_k = \delta_k(s)$ is the unit Dirac mass at the point $s = k$, then (0.6) reads

$$(W_\mu f)(x,t) = \sigma_{n-1} \sum_{k=0}^{\ell} (-1)^k \binom{\ell}{k} (1-kt)_+^{n/2-1} (M_{\sqrt{kt}} f)(x). \tag{0.10}$$

More general discrete measures (see [11], Sections 10.1, 10.2) can be also used.

Some comments are in order. The "usual" wavelet transform $f \to (f * g_t)(x)$ on $\mathbb{R}^n$, generated by the scaled version g_t of a radial wavelet function/measure g, can be "discovered", as above, starting from Riesz potentials $(I^\alpha f)(x) = c_{n,\alpha} \int_{\mathbb{R}^n} |x-y|^{\alpha-n} f(y) dy$, and using the generalization of Marchaud's method [11, p. 169]. These transforms provide a localization at a point (in accordance with the point singularity of the kernel of $I^\alpha f$). Similar spherical wavelet transforms (with a point localization) were introduced in [11]. In our case, which is typical for the integral geometrical setting, the localization is achieved in a neighborhood of a "big circle" representing the set of singularities of the kernel $|xy|^{\alpha-1}$. Our "wavelet transform" is just a tool, which enables us to build analytic continuation of $T^\alpha f$ for Re $\alpha \leq 0$ in a "nice" form (see [14] for further examples).

THEOREM A (inversion of T^α): *Let* $\mathrm{Re}\,\alpha > 0$, $\alpha \neq 1,3,5,\dots$, $\beta = (n+\alpha-1)/2$. *Assume that* μ *is a finite Borel measure on* $[0,\infty)$ *such that*

$$\int_0^\infty s^j d\mu(s) = 0 \quad \textit{for all } j = 0,1,\dots,[\mathrm{Re}\,\beta], \tag{0.11}$$

$$\int_0^\infty s^\gamma d|\mu|(s) < \infty \quad \textit{for some } \gamma > \mathrm{Re}\,\beta. \tag{0.12}$$

(i) *If* $\varphi = T^\alpha f$, $f \in L^p$, $1 \le p < \infty$, *then*

$$\int_0^\infty (W_\mu\varphi)(x,t)\frac{dt}{t^{1+(n+\alpha-1)/2}} \equiv \lim_{\varepsilon\to 0}\int_\varepsilon^\infty (W_\mu\varphi)(x,t)\frac{dt}{t^{1+(n+\alpha-1)/2}} = c_{\alpha,\mu}f(x) \tag{0.13}$$

where $\lim = \overset{(L^p)}{\lim} = \overset{a.e.}{\lim}$ *and* $c_{\alpha,\mu}$ *is defined by*

$$c_{\alpha,\mu} = \frac{2\pi^{n/2}}{\Gamma((n+\alpha)/2)}\begin{cases}\Gamma(-\beta)\displaystyle\int_0^\infty s^\beta d\mu(s) & \textit{if } \beta \neq 0,1,2,\dots,\\[2ex] \dfrac{(-1)^{\beta+1}}{\beta!}\displaystyle\int_0^\infty s^\beta \log s\, d\mu(s) & \textit{otherwise.}\end{cases} \tag{0.14}$$

(ii) *If* $f \in C$, *then* (0.13) *holds with* $\lim = \overset{(C)}{\lim}$.

Example: Consider the integral equation $\int_{\Sigma_n} |xy| f(y)dy = \varphi(x)$ with the cosine transform in the left-hand side [3, p. 379]. By Theorem A (for $\alpha = 2$), $f(x) = c_\mu^{-1}\int_0^\infty (W_\mu\varphi)(x,t)dt/t^{(n+3)/2}$, provided that

$$c_\mu = \frac{2\pi^{n-1/2}}{\Gamma(1+n/2)}\begin{cases}\dfrac{\pi(-1)^{n/2}}{\Gamma((n+3)/2)}\displaystyle\int_0^\infty s^{(n+1)/2} d\mu(s) & \text{if } n \text{ is even,}\\[2ex] \dfrac{(-1)^{(n+1)/2}}{((n+1)/2)!}\displaystyle\int_0^\infty s^{(n+1)/2}\log s d\mu(s) & \text{if } n \text{ is odd,}\end{cases}$$

$c_\mu \neq 0$, and μ satisfies (0.11)–(0.12) with $\beta = (n+1)/2$.

Our next result concerns the wavelet representation of $T^\alpha f, f \in L^p$, in the case $\mathrm{Re}\,\alpha \le 0$, when (0.2) fails, but T^α is still bounded in L^p. Assume that

$(Wf)(x,t)$ is the wavelet transform (0.7) with w represented by the fractional integral $w(s) = (I^{\theta}_{+}\mu_0)(s) = (1/\Gamma(\theta)) \int_0^s (s-t)^{\theta-1} d\mu_0(t), \quad \theta \geq 0$, where μ_0 is a finite Borel measure. Let

$$\int_0^\infty s^j d\mu_0(s) = 0 \quad \forall j = 0, 1, \dots, m;$$

$$m = \begin{cases} [-\operatorname{Re}\alpha/2+\theta] & \text{if } \{-\operatorname{Re}\alpha/2+\theta\} < 1/2, \\ [-\operatorname{Re}\alpha/2+\theta]+1 & \text{if } \{-\operatorname{Re}\alpha/2+\theta\} \geq 1/2; \end{cases}$$

$$\int_0^\infty s^\gamma d|\mu_0|(s) < \infty,$$

$$\text{for some } \gamma > \begin{cases} -\operatorname{Re}\ \alpha/2+\theta+1/2 & \text{if } \{-\operatorname{Re}\alpha/2+\theta\} < 1/2, \\ -\operatorname{Re}\alpha/2+\theta+1 & \text{if } \{-\operatorname{Re}\alpha/2+\theta\} \geq 1/2; \end{cases}$$

$$d_{\alpha,\mu} = \frac{2\pi^{n/2}}{\Gamma((1-\alpha)/2)} \begin{cases} \Gamma(\alpha/2) \int_0^\infty s^{-\alpha/2} w(s)ds & \text{if } -\alpha/2 \neq 0, 1, 2, \dots, \\ \dfrac{(-1)^{1-\alpha/2}}{(-\alpha/2)!} \int_0^\infty s^{-\alpha/2} w(s) \log s \, ds & \text{otherwise.} \end{cases}$$

THEOREM B: *Let* $(1-n)/2 + |1/p - 1/2|(n-1) \leq \operatorname{Re}\alpha \leq 0$, $1 < p < \infty$, $\theta \geq 1 + \operatorname{Re}\alpha/2 + [(\operatorname{Re}\alpha + n - 1)/2]$. *If* $d_{\alpha,\mu} \neq 0$, *then for* $f \in L^p$,

$$T^\alpha f = \frac{1}{d_{\alpha,\mu}} \int_0^\infty \frac{(Wf)(x,t)}{t^{1-\alpha/2}} dt$$

$$\equiv \lim_{\varepsilon \to 0} \frac{1}{d_{\alpha,\mu}} \int_\varepsilon^\infty \frac{(Wf)(x,t)}{t^{1-\alpha/2}} dt \quad \textit{in the } L^p\textit{-norm and a.e.}$$

COROLLARY C (representation of the spherical Radon transform; cf. [12, Th. 1.2]): *Let* $\theta = 1 + [(n-1)/2]$. *Assume that* $w(s)$ *is a* $\theta - 1$ *times continuously differentiable function on* $[0, \infty)$ *such that* $w^{(\theta-1)}(s)$ *is absolutely continuous on* $[0, \infty)$. *Moreover, let*

(a) $w^{(k)}(0) = 0$, $w^{(k)}(s) = o(s^{-k-1})$ *as* $s \to \infty$; $k = 0, 1, \dots, \theta - 1$;

(b) $\int_0^\infty w(s)ds = 0$;

(c) $\int_1^\infty s^\gamma |w^{(\theta)}(s)| ds < \infty$ *for some* $\gamma > \theta + 1/2$;

(d) $\varkappa_{n,w} = (2\pi^{n/2}/\Gamma(n/2)) \int_0^\infty w(s) \log(1/s) ds \neq 0$.

Then for $f \in L^p$, $1 < p < \infty$, $(Rf)(x) = \lim_{\varepsilon \to 0} \varkappa_{n,w}^{-1} \int_\varepsilon^\infty (Wf)(x,t)dt/t$ *in the* L^p*-norm and a.e.*

The paper is organized as follows. Sections 1 and 2 contain preliminaries and basic properties of (0.2). Section 3 is devoted to relations which link up wavelet transforms with $T^\alpha f$, $f \in C^\infty$. In Section 4 we prove Theorem A and characterize the range of T^α, $\mathrm{Re}\,\alpha > 0$, on functions f belonging to L^p, C, and on finite Borel measures. Apart from Theorem A, the main results are stated in Theorems 4.4 and 4.5. Section 5 contains the proof of Theorem B and an analogue of Theorem A for $\mathrm{Re}\,\alpha \le 0$.

ACKNOWLEDGEMENT: The author is grateful to the referee for valuable remarks improving the original text of the paper.

1. Preliminaries

Notation: Σ_n is the unit sphere in $\mathbb{R}^{n+1}$, $n \ge 2$;

$$\sigma_n = |\Sigma_n| = 2\pi^{(n+1)/2}/\Gamma((n+1)/2).$$

We denote by $\{Y_{j,k}(x)\}$, $x \in \Sigma_n$, the orthonormal basis of spherical harmonics on Σ_n. Here $j \in \mathbb{Z}_+ = \{0,1,2,\dots\}$; $k = 1,2,\dots,d_n(j)$ where $d_n(j)$ is the dimension of the subspace of spherical harmonics of degree j. The notation $L^p = L^p(\Sigma_n)$, $C = C(\Sigma_n)$, $C^\infty = C^\infty(\Sigma_n)$ is standard. The Fourier–Laplace decomposition of $f \in C^\infty$ is written as $f = \sum_{j,k} f_{j,k} Y_{j,k}$ (for more information about analysis on Σ_n see [11, 15] and references therein). Apart from the Jacobi polynomials $P_j^{(\alpha,\beta)}(\tau)$ and the Gegenbauer polynomials $C_j^{(n-1)/2}(\tau)$, we will use

$$H_j(\tau) = (\Gamma(j+1)\,\Gamma(n-1)/\Gamma(j+n-1))\, C_j^{(n-1)/2}(\tau). \tag{1.1}$$

The following relations hold [2]:

$$|H_j(\tau)| \le 1, \quad H_j(1) = 1, \tag{1.2}$$

$$H_j(0) = \begin{cases} \dfrac{(-1)^{j/2}\,\Gamma((j+1)/2)\,\Gamma(n/2)}{\pi^{1/2}\,\Gamma((j+n)/2)} & \text{for } j \text{ even,} \\ 0 & \text{for } j \text{ odd.} \end{cases}$$

The Funk–Hecke formula [2] reads

$$\int_{\Sigma_n} a(xy) Y_j(y)dy = \lambda Y_j(x), \quad \lambda = \sigma_{n-1} \int_{-1}^1 a(\tau)(1-\tau^2)^{n/2-1} H_j(\tau)d\tau, \tag{1.3}$$

where Y_j is a spherical harmonic of degree j and xy is the usual inner product.

In the following $[a]$ designates the integer part of $a \in \mathbb{R}; \{a\} = a - [a] \in [0,1); a_+ = \max(a;0); \mathbb{R}_+ = [0,\infty)$. The abbreviations "$\lesssim$" and "$\simeq$" indicate "$\leq$" and "$=$" if the latter hold up to a constant multiple.

LEMMA 1.1: *The mean value operator* (0.4) *enjoys the following properties:*

(a)

$$\sup_{\tau\in(-1,1)} \|M_\tau f\|_p \leq \|f\|_p, \quad f \in L^p, \ 1 \leq p \leq \infty. \tag{1.4}$$

(b) *For a spherical harmonic* $Y_j(x)$ *of degree* j,

$$(M_\tau Y_j)(x) = H_j(\tau) Y_j(x). \tag{1.5}$$

(c) *If* $f \in C^\infty(\Sigma_n)$, *then* $(M_\tau f)(x) \in C^\infty([-1,1])$ *in the* τ*-variable for each* $x \in \Sigma_n$. *If, moreover,* f *is even, then* $(M_\tau f)(x)$ *is an infinitely differentiable function of* τ^2.

The statements (a) and (b) are known. The first statement in (c) follows from $(M_\tau f)(x) = \sum_{j,k} H_j(\tau) f_{j,k} Y_{j,k}(x)$ because $|Y_{j,k}(x)| = o(j^{n/2-1})$, $f_{j,k} = o(j^{-m})$, $j \to \infty$, for all $m > 0$. The second statement in (c) is clear, since $f_{j,k} = 0$ for j odd, and $H_j(\tau)$ with j even is a polynomial of τ^2.

The next statement concerns spherical convolutions of the form

$$(K_\varepsilon f)(x) = \int_{\Sigma_n} k_\varepsilon(xy) f(y)dy, \tag{1.6}$$

$$k_\varepsilon(\tau) = \frac{(1-\tau^2)^{1-n/2}}{\varepsilon} k\left(\frac{1-\tau^2}{\varepsilon}\right), \quad \varepsilon > 0.$$

LEMMA 1.2 ([12]): *Let* f *be an even measurable function on* Σ_n, $(K^* f)(x) = \sup_{\varepsilon>0} |(K_\varepsilon f)(x)|$. *If* $k(s)$ *has a decreasing integrable majorant, then* $K^* f \lesssim f^*$, *where*

$$f^*(x) = \sup_{\tau\in(-1,1)} \frac{1}{|\sigma_\tau(x)|} \int_{\sigma_\tau(x)} |f(y)|dy, \quad \sigma_\tau(x) = \{y \in \Sigma_n : xy > \tau\}. \tag{1.7}$$

We will need the Riemann–Liouville fractional integrals [11]

(1.8)

$$(I_+^\lambda \nu)(s) = \frac{1}{\Gamma(\lambda)} \int_0^s (s-t)^{\lambda-1} d\nu(t), \quad (I_{1-}^\lambda \psi)(\tau) = \frac{1}{\Gamma(\lambda)} \int_\tau^1 (t-\tau)^{\lambda-1} \psi(\tau) d\xi.$$

Here $\operatorname{Re}\lambda > 0$, ν is a Borel measure on $\mathbb{R}_+$, $\psi(\tau)$ is a function on $(-1,1)$.

LEMMA 1.3: *Let* $\lambda' = \mathrm{Re}\,\lambda \geq 0$, $k \in \mathbb{Z}_+$,

$$\int_0^\infty s^j d\nu(s) = 0 \quad \text{for all } j = 0, 1, \ldots, m = \begin{cases} [\lambda'] + k & \text{if } \lambda \notin \mathbb{Z}_+, \\ \lambda & \text{otherwise}; \end{cases} \tag{1.9}$$

$$\int_1^\infty s^\gamma d|\nu|(s) < \infty \quad \text{for some } \gamma > \lambda' + k. \tag{1.10}$$

Then

(i)

(1.11)
$$(I_+^{1+\lambda}\nu)(s) = \begin{cases} O(s^{\lambda'}) & \text{if } 0 < s < 1, \\ O(s^{-k-\delta}),\ \delta = \min(\gamma - \lambda' - k, 1 - \{\lambda'\}), & \text{if } s \geq 1, \end{cases}$$

(ii)

$$\int_0^\infty (I_+^{1+\lambda}\nu)(s)\frac{ds}{s} = \begin{cases} \Gamma(-\lambda)\displaystyle\int_0^\infty s^\lambda d\nu(s) & \text{if } \lambda \notin \mathbb{Z}_+, \\ \dfrac{(-1)^{\lambda+1}}{\lambda!}\displaystyle\int_0^\infty s^\lambda \log s\, d\nu(s) & \text{if } \lambda \in \mathbb{Z}_+. \end{cases} \tag{1.12}$$

Proof: (i) We have $(I_+^{1+\lambda}\nu)(s) = (\int_0^{s/2} + \int_{s/2}^s)(\ldots) = g(s) + h(s)$ where, by (1.10), $|h(s)| \lesssim s^{\lambda'} \int_{s/2}^s d|\nu|(t) \lesssim s^{\lambda'-\gamma} = s^{-k-(\gamma-\lambda'-k)}$. In order to estimate $g(s)$, let

(1.13)
$$\frac{(s-t)^\lambda}{\Gamma(\lambda+1)} = \sum_{j=0}^m \frac{(-t)^j}{j!}\frac{s^{\lambda-j}}{\Gamma(\lambda+1-j)} + \frac{(-1)^{m+1}}{m!\,\Gamma(\lambda-m)}\int_0^t (t-\eta)^m (s-\eta)^{\lambda-m-1} d\eta$$

(for $\lambda \in \mathbb{Z}_+$ the integral term disappears). Then $g(s) = \sum_{j=0}^{m=1} c_j g_j(s)$,

$$g_j(s) = s^{\lambda-j}\int_0^{s/2} t^j d\nu(t),\ j = 1, \ldots, m;$$

$$g_{m+1}(s) = \int_0^{s/2} d\nu(t)\int_0^t (t-\eta)^m (s-\eta)^{\lambda-m-1} d\eta,$$

c_j $(j = 0, 1, \ldots, m+1)$ being the corresponding coefficients. For $j \leq m$ the relations (1.9) and (1.10) yield $|g_j(s)| = s^{\lambda'-j}|\int_{s/2}^\infty t^j d\nu(t)| \lesssim s^{\lambda'-\gamma}$. The term $g_{m+1}(s)$ can be estimated by making use of the formulae 2.12(1) and 2.9(3) from [2]:

$$|g_{m+1}(s)| \lesssim s^{\lambda'-m-1}\int_0^{s/2} t^{m+1} F(m+1-\lambda', 1; m+2; t/s)\, d|\nu|(t). \tag{1.14}$$

If $\lambda' > 0$, then according to 2.8(46) from [2],

$$|g_{m+1}(s)| \lesssim s^{\lambda'-m-1}\Big(\int_0^{1/2} + \int_{1/2}^{s/2}\Big)t^{m+1}d|\nu|(t) \tag{1.15}$$

$$\overset{(1.10)}{\lesssim} s^{\lambda'-\min(\gamma,m+1)} = s^{-\lambda-\delta},$$

$\delta = \min(\gamma - \lambda' - k, 1 - \{\lambda'\}) \in (0,1]$. If $\lambda' = 0$, then (1.14) yields

$$|g_{m+1}(s)| \lesssim \frac{m+1}{s^m}\int_0^{s/2} t^{m+1}d|\nu|(t)\int_0^1 \frac{\eta^m d\eta}{s-\eta t}$$

$$\lesssim \frac{1}{s^{m+1}}\int_0^{s/2} t^{m+1}d|\nu|(t) \lesssim s^{-k-\delta}$$

(cf. (1.15)). The second relation in (1.11) is proved. The first one is obvious.

(ii) Let us prove (1.12). By (1.11), $(I_+^{1+\lambda}\nu)(s)/s \in L^1(\mathbb{R}_+)$. Hence it suffices to find the limit $J_0 = \lim_{t\to 0} J(t)$ where $J(t) = \int_0^\infty e^{-ts}(I_+^{1+\lambda}\nu)(s)ds/s$. By changing the order of integration and using the formula $\int_u^\infty e^{-ts}(s-u)^\lambda ds/s = u^\lambda\Gamma(\lambda+1)\int_{ut}^\infty e^{-\eta}\eta^{-\lambda-1}d\eta$ [4], we get $J(t) = \int_t^\infty dv/v^{\lambda+1}\int_0^\infty e^{-uv}d\nu(u)$, $J_0 = \int_0^\infty dv/v^{\lambda+1}\int_0^\infty e^{-uv}d\nu(u)$. By (1.9),

$$J_0 = \int_0^\infty \frac{dv}{v^{\lambda+1}}\int_0^\infty \Big[e^{-uv} - \sum_{j=0}^m \frac{(-uv)^j}{j!}\Big]\, d\nu(u)$$

$$= \int_0^\infty d\nu(u)\int_0^\infty \Big[e^{-uv} - \sum_{j=0}^m \frac{(-uv)^j}{j!}\Big]\,\frac{dv}{v^{\lambda+1}},$$

and integration by parts leads to (1.14). ∎

2. Basic properties of the spherical fractional integrals

Assume that $T^\alpha f$ and Rf are defined by (0.2) and (0.1) respectively.

LEMMA 2.1: *Let* $\mathrm{Re}\,\alpha > 0$. *For a spherical harmonic* $Y_j(x)$ *of degree* j,

$$(T^\alpha Y_j)(x) = c_{j,\alpha}Y_j(x),\ c_{j,\alpha} = \begin{cases} (-1)^{j/2}\dfrac{\Gamma(j/2+(1-\alpha)/2)}{\Gamma(j/2+(n+\alpha)/2)} & \textit{if } j \textit{ is even,} \\ 0 & \textit{if } j \textit{ is odd.}\end{cases} \tag{2.1}$$

Proof: For j even the result follows from (1.3), (1.1), and the formula 2.21.2(5) from [10]. If j is odd, then (2.1) is obvious. ∎

For $\mathrm{Re}\,\alpha > 0$, T^α is bounded in L^p, $1 \le p \le \infty$. By (1.2), (0.3), and (2.1),

$$R = M_0 = \pi^{-1/2}\Gamma(n/2)\, T^0. \tag{2.2}$$

Hence, by (1.4), all operators in (2.2) are bounded in L^p, $1 \le p \le \infty$.

LEMMA 2.2: *If $f \in C^\infty(\Sigma_n)$, then $T^\alpha f$ can be extended to all $\alpha \in \mathbb{C}$ as a meromorphic function of α with simple poles at the points $\alpha = 1, 3, 5, \dots$.*

Proof: Let $f = f^+ + f^-$, $f^\pm(x) = (f(x) \pm f(-x))/2$. Then $T^\alpha f = T^\alpha f^+ = \gamma_{n,\alpha}\sigma_{n-1}\int_{-1}^1 |t|^{\alpha-1}(1-t^2)^{n/2-1}M_t f^+ dt$. Hence

$$T^\alpha f = \frac{\pi^{-n/2}\sigma_{n-1}\Gamma((1-\alpha)/2)}{2\Gamma(\alpha/2)}\int_0^1 \tau^{\alpha/2-1}(1-\tau)^{n/2-1}M_{\sqrt{\tau}}f^+ d\tau, \qquad \operatorname{Re}\alpha \in (0,1), \tag{2.3}$$

and the result becomes clear due to Lemma 1.1 (c). ∎

In the following the notation T^α will also be used for $\operatorname{Re}\alpha \le 0$. Thus,

$$(T^\alpha f)(x) = \sum_{j,k} c_{j,\alpha} f_{j,k} Y_{j,k}(x), \quad f \in C^\infty, \quad \alpha \in \mathbb{C} \quad (\alpha \ne 1, 3, 5, \dots), \tag{2.4}$$

$c_{j,\alpha} = (-1)^{j/2}\Gamma(j/2 + (1-\alpha)/2)/\Gamma(j/2 + (n+\alpha)/2)$ for j even and $c_{j,\alpha} = 0$ for j odd.

LEMMA 2.3: *If $\alpha \notin \{1, 3, 5, \dots\}$, then T^α: $C^\infty \to C^\infty_{\text{even}}$ is a linear continuous map. If $\alpha \notin \{1, 3, 5, \dots\} \cup \{-n, -n-2, -n-4, \dots\}$, then T^α is an automorphism of C^∞_{even} and*

$$(T^\alpha)^{-1} = T^{1-n-\alpha}. \tag{2.5}$$

This statement follows immediately from (2.4) because $c_{j,\alpha} = O(j^{(1-n-2\alpha)/2})$ as $j \to \infty$.

For $\operatorname{Re}\alpha \le 0$, the behaviour of $T^\alpha f$, $f \in L^p$, is rather delicate. In order to make it clear we consider the more general operator family defined on $f \in C^\infty$ by

$$(A^\alpha f)(x) = \sum_{j,k} i^j \frac{\Gamma(j/2 + (1-\alpha)/2)}{\Gamma(j/2 + (n+\alpha)/2)} f_{j,k} Y_{j,k}(x), \quad \alpha \in \mathbb{C}; \alpha \ne 1, 3, 5, \dots \tag{2.6}$$

(see [15]). The latter coincides with $T^\alpha f$ for f even. Given $\gamma \in \mathbb{R}$ and $p \in (1, \infty)$, let $L^\gamma_p = L^\gamma_p(\Sigma_n)$ be the Sobolev space, which consists of distributions with the property: for each $f \in L^\gamma_p$ there is a function $f^{(\gamma)} \in L^p$ such that $f^{(\gamma)}_{j,k} = (j+1)^\gamma f_{j,k}$ for all Fourier–Laplace coefficients. We put $\|f\|_{L^\gamma_p} = \|f^{(\gamma)}\|_p$.

THEOREM 2.4: *Let $1 < p < \infty$, $\alpha \in \mathbb{C}$; $\alpha \neq 1, 3, 5, \dots$.*

(i) *The operator (2.4) can be extended as a linear bounded operator, acting from L_p^β into L_p^γ provided*

$$\text{(2.7)} \qquad \operatorname{Re}\alpha \geq \gamma - \beta - \frac{n-1}{2} + \left|\frac{1}{p} - \frac{1}{2}\right|(n-1).$$

(ii) *If (2.7) fails, then there is an even function $f_0 \in L_p^\beta$ such that $T^\alpha f_0 \notin L_p^\gamma$.*

Proof: Let $f = f^+ + f^-$, $f^\pm(x) = (f(x) \pm f(-x))/2$. Then $T^\alpha f = T^\alpha f^+ = A^\alpha f^+$, $\|f^+\|_{L_p^\beta} \leq \|f\|_{L_p^\beta}$. The estimate $\|A^\alpha f\|_{L_p^\gamma} \lesssim \|f\|_{L_p^\beta}$ is equivalent to $\|A^1 f\|_{L_p^\delta} \lesssim \|f\|_p$, $\delta = \gamma - \beta - \text{Re } \alpha + 1$. This can be easily checked by using the Strichartz multiplier theorem [17]. The above estimate of $A^1 f$ holds if and only if (2.7) is satisfied [7]. In order to prove (ii) it suffices to reproduce the argument from [7, Section 5] for the function $f_0(x) = (I - \Delta_\Sigma)^{-\beta/2}[F_\varepsilon(x_{n+1}) + F_\varepsilon(-x_{n+1})]$ where Δ_Σ is the Beltrami–Laplace operator on Σ_n and F_ε is defined by the equality (52) (or (54)) from [7]. ∎

By making use of the argument from [7, 9] it is not difficult to obtain sharp conditions under which T^α is bounded from L_p^β into L_q^γ with $q \geq p$.

Denote by L^p_{even} and $L^\gamma_{p,\text{even}}$ the spaces of even functions (or distributions), belonging to L^p and L_p^γ respectively, with usual norms.

COROLLARY 2.5: $L^\delta_{p,\text{even}} \subset T^\alpha(L^p_{\text{even}}) \subset L^\gamma_{p,\text{even}}$, *provided*

$$\text{(2.8)} \quad \gamma = \text{Re } \alpha + \frac{n-1}{2} - \left|\frac{1}{p} - \frac{1}{2}\right|(n-1), \quad \delta = \text{Re } \alpha + \frac{n-1}{2} + \left|\frac{1}{p} - \frac{1}{2}\right|(n-1),$$

$$\alpha \notin \{1, 3, 5, \dots\} \cup \{-n, -n-2, -n-4, \dots\}.$$

The right embedding follows from Theorem 2.4 with $\beta = 0$. If $f \in L^\delta_{p,\text{even}}$, then $f = T^\alpha T^{1-n-\alpha} f$ where $T^{1-n-\alpha} f \in L^p$ (use Theorem 2.4 with $\beta = \delta$ and $\gamma = 0$).

By Corollary 2.5 and Theorem 2.4, it is impossible to characterize $T^\alpha(L^p)$ in terms of the Sobolev spaces for $p \neq 2$. We will do this later with the aid of wavelet transforms.

COROLLARY 2.6: *For $1 < p < \infty$, $\operatorname{Re}\alpha \leq 0$, T^α is bounded in L^p if and only if*

$$\text{(2.9)} \qquad \operatorname{Re}\alpha \geq \frac{1-n}{2} + \left|\frac{1}{p} - \frac{1}{2}\right|(n-1).$$

3. Spherical wavelet transforms and auxiliary relations for C^∞-functions

Since $T^\alpha f \equiv 0$ for f odd, in the following we deal with even functions f only and write C^∞ instead of C^∞_{even} (similarly for L^p and other spaces). It is convenient to deal with wavelet transforms of the form (0.6).

LEMMA 3.1: *Let $f \in L^p$, $1 \le p \le \infty$, $n \ge 2$.*

(i) *If μ is a finite Borel measure on $\mathbb{R}_+$, then*

$$\|W_\mu f\|_p \le 2\pi^{n/2} t^{n/2-1} \|f\|_p (I_+^{n/2}|\mu|)(1/t) \le 2\pi^{n/2} \|\mu\| \, \|f\|_p \tag{3.1}$$

where $\|\mu\|$ is the total variation of $|\mu|$.

(ii) *If $d\mu(s) = w(s)ds$ and $w = I_+^\theta \mu_0$, $\theta > 0$, for some finite Borel measure μ_0, then*

$$\|W_\mu f\|_p \le 2\pi^{n/2} t^{n/2-1} \|f\|_p (I_+^{n/2+\theta}|\mu_0|)(1/t) \le 2\pi^{n/2} t^{-\theta} \|\mu_0\| \, \|f\|_p. \tag{3.2}$$

Proof: By (1.4), from (0.6) we have

$$\begin{aligned} \|W_\mu f\|_p &\le \sigma_{n-1} \|f\|_p \int_0^{1/t} (1-ts)^{n/2-1} d|\mu|(s) \\ &= 2\pi^{n/2} t^{n/2-1} \|f\|_p (I_+^{n/2}|\mu|)(1/t) \end{aligned} \tag{3.3}$$

which gives (3.1). The statement (ii) is a consequence of (3.3). ∎

Due to (0.5) and (2.5), one can expect

$$f = c \int_0^\infty (W_\mu T^\alpha f)(x,t) \frac{dt}{t^{1+\beta}}, \quad \beta = (n+\alpha-1)/2, \tag{3.4}$$

for suitable μ and $c = c(\alpha, \mu)$. The precise sense to (3.4) will be given later. Now we start with some preparations. Consider the operator family

$$(M_t^\alpha f)(x) = \sum_{j,k} u_j^\alpha(t) f_{j,k} Y_{j,k}(x), \tag{3.5}$$

(3.6)
$$u_j^\alpha(t) = \frac{\Gamma((n+\alpha)/2)\ \Gamma(1+j/2)}{\Gamma((j+n+\alpha)/2)} (1-t)^{-(\alpha+1)/2} P_{j/2}^{((n+\alpha)/2-1,-(\alpha+1)/2)}(1-2t),$$

assuming $f \in C^\infty$, $0 \le t < 1$, $-n < \operatorname{Re}\alpha < 1$. We recall that f is even.

LEMMA 3.2: (i) *For each compact set K in the strip $-n < \operatorname{Re}\alpha < 1$ and $f \in C^\infty$, there is a constant $C_{K,f}$ such that*

$$\sup_x |(M_t^\alpha f)(x)| \le C_{K,f}(1-t)^{-(\operatorname{Re}\alpha+1)/2} \quad \forall\alpha \in K. \tag{3.7}$$

(ii)

$$\lim_{t\to 0}(M_t^\alpha f)(x) = f(x) \quad \text{uniformly on } \Sigma_n. \tag{3.8}$$

Proof: Owing to the formula 2.22.2(2) from [10], we have

$$P_{j/2}^{((n+\alpha)/2-1,-(\alpha+1)/2)}(s) = \frac{(s+1)^{(\alpha+1)/2}}{B(\sigma-(\alpha+1)/2, 1-\sigma+j/2)} \times \int_{-1}^{s} (\tau+1)^{-\sigma}(s-\tau)^{\sigma-1-(\alpha+1)/2} P_{j/2}^{(\sigma+(n-3)/2,-\sigma)}(\tau)d\tau$$

for each σ such that $1 > \sigma > (\operatorname{Re}\alpha+1)/2$. If $\sigma \ge (3-n)/4$, then [2, 10.18(12)]

$$\max_{-1\le\tau\le1} |P_{j/2}^{(\sigma+(n-3)/2,-\sigma)}(\tau)| = \binom{\sigma+(n-5+j)/2}{j/2}, \tag{3.9}$$

and therefore (one can assume $\sigma \ne \frac{1}{2} (\operatorname{mod} 1)$)

$$|P_{j/2}^{((n+\alpha)/2-1,-(\alpha+1)/2)}(s)| \le c_{\sigma,\alpha}\left|\frac{\Gamma((1-\alpha+j)/2)\ \Gamma(\sigma+(n-3+j)/2)}{\Gamma(1-\sigma+j/2)\ \Gamma(1+j/2)}\right|,$$

$$c_{\sigma,\alpha} = \frac{\Gamma(1-\sigma)\ \Gamma(\sigma-(\operatorname{Re}\ \alpha+1)/2)}{|\Gamma(\sigma-(\alpha+1)/2)|\ \Gamma((1-\operatorname{Re}\alpha)/2)\ \Gamma(\sigma+(n-3)/2)}.$$

Due to the properties of Γ-functions [11, p. 390] it follows that for each compact set K in the strip $-n < \operatorname{Re}\ \alpha < 1$ there exists a constant C_K such that

$$|u_j^\alpha(t)| \le C_K j^{-\operatorname{Re}\alpha}(1-t)^{-(\operatorname{Re}\alpha+1)/2} \quad \forall\alpha\in K. \tag{3.10}$$

This implies (i). The second statement is clear, because $u_j^\alpha(0) = 1$ (see [2, 10.8(3)]). ∎

LEMMA 3.3: *Let $f \in C^\infty$, $1-n < \operatorname{Re}\alpha < 1$, $\beta = (n+\alpha-1)/2$, $n \ge 2$. Then*

$$\frac{\Gamma((n+\alpha)/2)}{\Gamma(n/2)}(1-t)^{n/2-1} M_{\sqrt{t}} T^\alpha f = (I_{1-}^\beta M_{(\cdot)}^\alpha f)(t),\ t\in[0,1), \tag{3.11}$$

where $M_{\sqrt{t}}$ and I^{β}_{1-} are defined by (0.4) *and* (1.8) *respectively.*

Proof: It suffices to prove (3.11) for spherical harmonics $f = Y_j$ of even degree j. By (1.5), (2.4) and (3.5), the equality (3.11) reads

$$c_{j,\alpha}\frac{\Gamma((n+\alpha)/2)}{\Gamma(n/2)}(1-s)^{n/2-1}H_j(\sqrt{s}) = (I^{\beta}_{1-}u^{\alpha}_j)(s), \quad 0 \le s < 1. \tag{3.12}$$

Owing to the formulae 3.15.1(5) and 10.8(16) from [2], we have

$$\frac{1}{\Gamma(n/2)}H_j(\sqrt{s}) = \frac{(-1)^{j/2}\Gamma(1+j/2)}{\Gamma((j+n)/2)}P^{(-1/2,n/2-1)}_{j/2}(1-2s). \tag{3.13}$$

Thus, the left-hand side of (3.12) has the form

$$c_{j,\alpha}(-1)^{j/2}\frac{\Gamma(1+j/2)\ \Gamma((n+\alpha)/2)}{\Gamma((j+n)/2)}(1-s)^{n/2-1}P^{(-1/2,n/2-1)}_{j/2}(1-2s).$$

By [10, 2.22.2(2)] this coincides with the right-hand side of (3.12). ∎

Now we pass to justification of the inversion formula (3.4) for $f \in C^{\infty}$. Denote

$$(\mathcal{T}^{\alpha}_{\varepsilon}\varphi)(x) = \int_{\varepsilon}^{\infty}(W_{\mu}\varphi)(x,t)\frac{dt}{t^{1+\beta}}, \quad \beta = (n+\alpha-1)/2, \tag{3.14}$$

and assume (0.11) and (0.12) for $1-n < \mathrm{Re}\,\alpha < 1$. In the case $\mathrm{Re}\,\alpha = 1-n$ we suppose

$$d\mu(s) = w(s)ds, \quad w = I^{\theta}_{+}\mu_0 \quad \text{for some } \theta > 0, \tag{3.15}$$

$$\int_0^{\infty} s^j d\mu_0(s) = 0 \quad \text{for all } j = 0,1,\dots,[\mathrm{Re}\ \beta+\theta], \tag{3.16}$$

$$\int_0^{\infty} s^{\gamma_0} d|\mu_0|(s) < \infty \quad \text{for some } \gamma_0 > \mathrm{Re}\ \beta+\theta. \tag{3.17}$$

Remark 3.4: For short, sometimes we write $\mu = I^{\theta}_{+}\mu_0$ in both cases. If $\theta = 0$, it means that $\mu = \mu_0$, and for $\theta > 0$ this equality is understood as (3.15). In particular, one can assume θ to be an integer and $w(s)$ to be such that $w(0) = w'(0) = \cdots = w^{(\theta-1)}(0) = 0$ with $w^{(\theta)}(s)$ satisfying (3.16), (3.17).

By Lemma 3.1, for $\varphi \in L^p$, $p \in [1,\infty]$, we have

$$\|\mathcal{T}^{\alpha}_{\varepsilon}\varphi\|_p \le c\varepsilon^{-\beta'-\theta}\|\varphi\|_p, \quad \beta' = \mathrm{Re}\ \beta, \quad \theta \ge 0. \tag{3.18}$$

LEMMA 3.5: *Let $f \in C^\infty$, $1-n \le \operatorname{Re}\alpha < 1$, $n \ge 2$, $\beta = (n+\alpha-1)/2$. Assume that μ is chosen according to (0.11)–(0.12) and (3.15)–(3.17). Then*

$$(3.19)\qquad \mathcal{T}_\varepsilon^\alpha T^\alpha f = \int_0^{1/\varepsilon} \lambda_\alpha(s) M_{\varepsilon s}^\alpha f ds, \quad \lambda_\alpha(s) = \frac{2\pi^{n/2}}{s\,\Gamma((n+\alpha)/2}(I_+^{1+\beta}\mu)(s),$$

$$(3.20)\qquad \lambda_\alpha \in L^1(\mathbb{R}_+), \quad \int_0^\infty \lambda_\alpha(s)ds = c_{\alpha,\mu},$$

where $c_{\alpha,\mu}$ is defined by (0.14).

Proof: Let first $1-n < \operatorname{Re}\alpha < 1$. The relations (0.6) and (3.11) yield

$$(3.21)\qquad (W_\mu T^\alpha f)(x,t) = \frac{2\pi^{n/2}t^\beta}{\Gamma((n+\alpha)/2)}\int_0^{1/t}(I_+^\beta\mu)(\xi)M_{t\xi}^\alpha f\, d\xi.$$

Indeed, by putting $g(\tau) = M_\tau^\alpha f$ we have

$$\begin{aligned} W_\mu T^\alpha f &= \frac{2\pi^{n/2}}{\Gamma((n+\alpha)/2)\,\Gamma(\beta)}\int_0^{1/t}d\mu(s)\int_0^{1-ty}\tau^{\beta-1}g(ts+\tau)d\tau \\ &= \frac{2\pi^{n/2}t^\beta}{\Gamma((n+\alpha)/2)\,\Gamma(\beta)}\int_0^{1/t}d\mu(s)\int_s^{1/t}(\xi-s)^{\beta-1}g(t\xi)\,d\xi \\ &= \frac{2\pi^{n/2}t^\beta}{\Gamma((n+\alpha)/2)}\int_0^{1/t}g(t\xi)(I_+^\beta\mu)(\xi)d\xi. \end{aligned}$$

We note that $I_{0+}^\beta\mu \in L^1(\mathbb{R}_+)$ (see Corollary 4.13′ from [11]). Furthermore,

$$\mathcal{T}_\varepsilon^\alpha T^\alpha f = \frac{2\pi^{n/2}}{\Gamma((n+\alpha)/2)}\int_0^1 g(u)\frac{du}{u}\int_0^{u/\varepsilon}(I_+^\beta\mu)(\eta)d\eta = \int_0^{1/\varepsilon} g(\varepsilon s)\;\lambda_\alpha(s)\;ds.$$

The change of the order of integration can be justified by using (3.7). The relations (3.20) and (0.14) are implied by Lemma 1.3. The validity of (3.19) for Re $\alpha = 1-n$ follows by analytic continuation (use Lemma 3.1 and Lemma 3.2(i)). If $w = I_+^\theta\mu_0$, then, owing to (3.16) and (3.17), by Corollary 4.13′ from [11] we have $\int_0^\infty w(s)ds = 0$, and $\int_1^\infty s^\gamma|w(s)|ds < \infty$ for some $\gamma > 0$. By Lemma 1.3 these yield (3.20) and (0.14). ∎

THEOREM 3.6: *If μ satisfies (0.11)–(0.12) and (3.15)–(3.17), then for each $x \in \Sigma_n$,*

$$\lim_{\varepsilon\to 0}\int_\varepsilon^\infty (W_\mu T^\alpha f)(x,t)\frac{dt}{t^{1+(n+\alpha-1)/2}} = c_{\alpha,\mu}f(x), \quad f \in C^\infty,\ 1-n \le \operatorname{Re}\alpha < 1,$$

where $c_{\alpha,\mu}$ is the constant (0.14).

Proof: One has to check the equality $\lim_{\varepsilon\to 0} \mathcal{T}_\varepsilon^\alpha T^\alpha f = c_{\alpha,\mu} f$. Due to (3.19),

$$\mathcal{T}_\varepsilon^\alpha T^\alpha f = \Big(\int_0^{1/2\varepsilon} + \int_{1/2\varepsilon}^{1/\varepsilon}\Big)\lambda_\alpha(s) M_{\varepsilon s}^\alpha f \, ds = A_{\varepsilon,1}^\alpha f + A_{\varepsilon,2}^\alpha f. \tag{3.22}$$

By (3.7), (3.8) and (3.20), we get $\lim_{\varepsilon\to 0} A_1^\alpha f = c_{\alpha,\mu} f$. The term $A_{\varepsilon,2}^\alpha f$ tends to 0, because by (1.13) and (3.7), $|A_{\varepsilon,2}^\alpha f| \lesssim \int_{1/2\varepsilon}^{1/\varepsilon}(1-\varepsilon s)^{-(\alpha'+1)/2} s^{-\delta-1} ds = O(\varepsilon^\delta)$, $\delta > 0$, $\alpha' = \text{Re }\alpha$. ∎

4. L^p-theory (the case $\text{Re}\,\alpha > 0$)

LEMMA 4.1 (an integral representation of (3.5)): *Let $\text{Re}\,\alpha > 0$, $f \in C^\infty$. Then*

$$(M_t^\alpha f)(x) = \int_{\Sigma_n} k_t^\alpha(xy) f(y) dy, \tag{4.1}$$

$$k_t^\alpha(\tau) = \frac{\Gamma((n+\alpha)/2)}{2\pi^{n/2}\Gamma(\alpha/2)}(1-t)^{-(\alpha+1)/2} t^{1-(\alpha+n)/2} |\tau| (t-1+\tau^2)_+^{\alpha/2-1}. \tag{4.2}$$

Proof: According to the Funk–Hecke formula (1.3) it suffices to show that

$$\frac{\sigma_{n-1}\Gamma((n+\alpha)/2)}{2\pi^{n/2}\Gamma(\alpha/2)}(1-t)^{-(\alpha+1)/2} t^{1-(\alpha+n)/2} \times \int_{-1}^1 |\tau|(t-1+\tau^2)_+^{\alpha/2-1}(1-\tau^2)^{n/2-1} H_j(\tau) d\tau = u_j(t)$$

(see (3.6)). Put $\tau^2 = s$, $1 - t = u$. Then the above relation can be checked by using (3.13) and the formula 2.22.2(7) from [10]. ∎

By analyticity, (3.19) can be extended to all $\text{Re}\,\alpha > 0$ ($\alpha \neq 1, 3, 5, \dots$). Below we construct this analytic continuation and show its convergence as $\varepsilon \to 0$.

LEMMA 4.2: *Let $\text{Re}\,\alpha > 0$, $\alpha \neq 1, 3, 5, \dots$. Assume that $f \in C^\infty$ and μ satisfies* (0.11), (0.12). *Then there exist spherical convolution operators $A_{\varepsilon,1}^\alpha$ and $A_{\varepsilon,2}^\alpha$ such that*

$$\mathcal{T}_\varepsilon^\alpha T^\alpha f = A_{\varepsilon,1}^\alpha f + A_{\varepsilon,2}^\alpha f, \quad 0 < \varepsilon < 1/2, \tag{4.3}$$

and the following assertions hold:

(a)

$$\sup_{0<\varepsilon<1/2} |A^{\alpha}_{\varepsilon,1} f| \le c_1 f^*, \quad \|A^{\alpha}_{\varepsilon,1} f\|_p \le c_2 \|f\|_p \quad \forall p \in [1,\infty], \tag{4.4}$$

where f^* is the maximal function (1.7) *and c_2 is independent of ε.*

(b) *For each spherical harmonic Y_j of even degree j,*

$$\lim_{\varepsilon\to 0} A^{\alpha}_{\varepsilon,1} Y_j = c_{\alpha,\mu} Y_j, \quad c_{\alpha,\mu} \text{ being defined by (0.14).} \tag{4.5}$$

(c)

$$\sup_x |(A^{\alpha}_{\varepsilon,2} f)(x)| \le c_3 \varepsilon^{\delta} \|f\|_1 \quad \text{for some } \delta > 0. \tag{4.6}$$

Proof: For $0 < \operatorname{Re}\alpha < 1$, the equality (4.3) is known in the form (3.22) with

$$A^{\alpha}_{\varepsilon,1} f = \int_0^{1/2\varepsilon} \lambda_\alpha(s) M^{\alpha}_{\varepsilon s} f \, ds, \qquad A^{\alpha}_{\varepsilon,2} f = \int_{1/2\varepsilon}^{1/\varepsilon} \lambda_\alpha(s) M^{\alpha}_{\varepsilon s} f \, ds. \tag{4.7}$$

By (4.1), we have $(A^{\alpha}_{\varepsilon,i} f)(x) = \int_{\Sigma_n} \Lambda^{\alpha}_{\varepsilon,i}(xy) f(y) dy$, $i = 1, 2$, where

$$\begin{aligned}(4.8)\,\Lambda^{\alpha}_{\varepsilon,1}(\tau) = &\frac{|\tau|\,\Gamma((n+\alpha)/2)}{2\pi^{n/2}\Gamma(\alpha/2)} \\ &\times \int_0^{1/2\varepsilon} \lambda_\alpha(s)(1-\varepsilon s)^{-(\alpha+1)/2} (\varepsilon s)^{1-(\alpha+n)/2} (\tau^2 - 1 + \varepsilon s)_+^{\alpha/2-1} ds,\end{aligned}$$

$$\begin{aligned}(4.9)\,\Lambda^{\alpha}_{\varepsilon,2}(\tau) = &\frac{|\tau|\,\Gamma(n+\alpha)/2)}{2\pi^{n/2}\Gamma(\alpha/2)} \\ &\times \int_{1/2\varepsilon}^{1/\varepsilon} \lambda_\alpha(s)(1-\varepsilon s)^{-(\alpha+1)/2} (\varepsilon s)^{1-(\alpha+n)/2} (\tau^2 - 1 + \varepsilon s)_+^{\alpha/2-1} ds.\end{aligned}$$

We regard (4.3) as the analytic continuation (a.c.) of (3.22) to $\{\alpha\colon \operatorname{Re}\alpha \ge 1\}$. Note that a.c.$\mathcal{T}^{\alpha}_{\varepsilon} T^{\alpha} f$ and a.c.$A^{\alpha}_{\varepsilon,1} f$ have the same form as for $0 < \operatorname{Re}\alpha < 1$. In order to get a.c.$A^{\alpha}_{\varepsilon,2} f$, one should transform (4.9). We proceed as follows.

STEP 1: Let us prove (4.4). For $0 < \varepsilon < 1/2$, by putting $\alpha' = \operatorname{Re}\,\alpha$ we have

$$\begin{aligned}|\Lambda^{\alpha}_{\varepsilon,1}(\tau)| &\lesssim |\tau| \Big(\int_0^1 + \int_1^{1/2\varepsilon} \Big) |\lambda_\alpha(s)| (1-\varepsilon s)^{-(\alpha'+1)/2} (\varepsilon s)^{1-(\alpha'+n)/2} \\ &\qquad\qquad \times (\tau^2 - 1 + \varepsilon s)_+^{\alpha'/2-1} ds \\ &= I_{\varepsilon,1}(\tau) + I_{\varepsilon,2}(\tau).\end{aligned}$$

It suffices to show that for some $\delta > 0$,

(4.10)
$$I_{\varepsilon,i}(\tau) \lesssim \frac{(1-\tau^2)^{1-n/2}}{\varepsilon} h\Big(\frac{1-\tau^2}{\varepsilon}\Big), \quad h(\eta) = \begin{cases} \eta^{\delta-1} & \text{if } \eta \le \varepsilon, \\ \eta^{-\delta-1} & \text{if } \eta > \varepsilon, \end{cases} \quad i = 1,2.$$

Indeed, the first inequality in (4.4) then follows by Lemma 1.2. The second one is a consequence of the simple estimate

$$\begin{aligned}\int_{-1}^{1} |\Lambda^{\alpha}_{\varepsilon,1}(\tau)|(1-\tau^2)^{n/2-1} d\tau &\lesssim \frac{1}{\varepsilon}\int_0^1 h\Big(\frac{1-\tau^2}{\varepsilon}\Big)d\tau \\ &= \Big(\int_0^{1/2\varepsilon} + \int_{1/2\varepsilon}^{1/\varepsilon}\Big)\frac{h(\eta)d\eta}{\sqrt{1-\varepsilon\eta}} \\ &\lesssim \int_0^\infty h(\eta)d\eta + \varepsilon^{\delta}.\end{aligned}$$

Denote $z = (1-\tau^2)/\varepsilon$ and consider $I_{\varepsilon,1}$. If $z > 1$, then $I_{\varepsilon,1}(\tau) \equiv 0$. In the case $z \le 1$ by (1.11) we have $|\lambda_\alpha(s)| \lesssim s^{\beta'-1} = s^{(n+\alpha'-3)/2}$, and therefore

$$\begin{aligned} I_{\varepsilon,1} &\lesssim \varepsilon^{-(\alpha'+n+1)/2} \int_z^1 \Big(\frac{1}{\varepsilon} - s\Big)^{-(\alpha'+1)/2} (s-z)^{\alpha'/2-1}\frac{ds}{s^{1/2}} \\ &= \frac{\varepsilon^{-(\alpha'+n+1)/2}}{z}\int_1^{1/z}\Big(\frac{1}{\varepsilon z} - u\Big)^{-(\alpha'+1)/2}(u-1)^{\alpha'/2-1}\frac{du}{u^{1/2}} \\ &\qquad (1/\varepsilon z - u > 1/\varepsilon z - 1/z > 1/2\varepsilon z) \\ &\lesssim \frac{z^{(\alpha'-1)/2}}{\varepsilon^{n/2}}\int_1^{1/z}(u-1)^{\alpha'/2-1}\frac{du}{u^{1/2}} \lesssim \frac{1}{\varepsilon^{n/2}}\left\{\begin{matrix} 1 & \text{if } \alpha' > 1, \\ 1 + |\log z| & \text{if } \alpha' = 1 \end{matrix}\right\} \\ &= \frac{(1-\tau^2)^{1-n/2}}{\varepsilon}\Big(\frac{1-\tau^2}{\varepsilon}\Big)^{n/2-1}\Big\{\dots\Big\} \\ &\lesssim \frac{(1-\tau^2)^{1-n/2}}{\varepsilon}\Big(\frac{1-\tau^2}{\varepsilon}\Big)^{\delta-1} \quad \forall\delta \in (0, n/2). \end{aligned}$$

Let us estimate $I_{\varepsilon,2}$. By (1.11), for some $\delta > 0$ as above we have

(4.11)
$$\begin{aligned} I_{\varepsilon,2} &\lesssim \frac{\varepsilon^{-(\alpha'+n+1)/2}}{z^{\delta+(\alpha'+n+1)/2}}\int_{1/z}^{1/2\varepsilon z}(u-1)_+^{\alpha'/2-1}\Big(\frac{1}{\varepsilon z} - u\Big)^{-(\alpha'+1)/2}\frac{du}{u^{\delta+(\alpha'+n)/2}} \\ &\lesssim \frac{\varepsilon^{-n/2}}{z^{\delta+n/2}}\int_{1/z}^{1/2\varepsilon z}(u-1)_+^{\alpha'/2-1}\frac{du}{u^{\delta+(\alpha'+n)/2}} \\ &\qquad \Big(\text{use the inequality } \frac{1}{\varepsilon z} - u > \frac{1}{2\varepsilon z}\Big).\end{aligned}$$

If $z \leq 1$, then

$$I_{\varepsilon,2} \lesssim \frac{\varepsilon^{-n/2}}{z^{\delta+n/2}} \int_{1/z}^{\infty} (u-1)^{\alpha'/2-1} \frac{du}{u^{\delta+(\alpha'+n)/2}}$$
$$\lesssim \varepsilon^{-n/2} = \frac{(1-\tau^2)^{1-n/2}}{\varepsilon} \Big(\frac{1-\tau^2}{\varepsilon}\Big)^{n/2-1}.$$

If $1 < z < 1/2\varepsilon$, then

$$I_{\varepsilon,2} \lesssim \frac{\varepsilon^{-n/2}}{z^{\delta+n/2}} \int_{1}^{\infty} \frac{(u-1)^{\alpha'/2-1} du}{u^{\delta+(\alpha'+n)/2}} = \text{const}\ \frac{(1-\tau^2)^{1-n/2}}{\varepsilon} \Big(\frac{1-\tau^2}{\varepsilon}\Big)^{-\delta-1}.$$

In the case $z \geq 1/2\varepsilon$ we have $I_{\varepsilon,2} \equiv 0$. Thus (4.4) is proved.

STEP 2: Let us check (4.6). It suffices to show that a.c.$\Lambda^{\alpha}_{\varepsilon,2} \lesssim \varepsilon^{\delta}$ uniformly in α for α belonging to arbitrary compact domain $G \subset \{\alpha\colon \operatorname{Re}\alpha > 0\}$.

We write $\Lambda^{\alpha}_{\varepsilon,2} = J^{\alpha}_{\varepsilon,1} + J^{\alpha}_{\varepsilon,2}$, where

$$J^{\alpha}_{\varepsilon,1} = \frac{|\tau|\ \Gamma((n+\alpha)/2)}{2\pi^{n/2}\Gamma(\alpha/2)}$$
$$\times \int_{1/2\varepsilon}^{(1-\tau^2/2)/\varepsilon} \lambda_\alpha(s)(1-\varepsilon s)^{-(\alpha+1)/2}(\varepsilon s)^{1-(\alpha+n)/2}(\tau^2-1+\varepsilon s)_+^{\alpha/2-1} ds, \tag{4.12}$$

$$J^{\alpha}_{\varepsilon,2} = \frac{|\tau|\ \Gamma((n+\alpha)/2)}{2\pi^{n/2}\Gamma(\alpha/2)}$$
$$\times \int_{(1-\tau^2/2)/\varepsilon}^{1/\varepsilon} \lambda_\alpha(s)(1-\varepsilon s)^{-(\alpha+1)/2}(\varepsilon s)^{1-(\alpha+n)/2}(\tau^2-1+\varepsilon s)^{\alpha/2-1} ds. \tag{4.13}$$

The first term is an analytic function of α for $\operatorname{Re}\alpha > 0$, and can be estimated as follows:

$$|J^{\alpha}_{\varepsilon,1}| \lesssim \varepsilon^{-(\alpha'+n+1)/2}|\tau| \int_{1/2\varepsilon}^{(1-\tau^2/2)/\varepsilon} \Big(\frac{1}{\varepsilon}-s\Big)^{-(\alpha'+1)/2} (s-z)_+^{\alpha'/2-1} \frac{ds}{s^{\delta+(\alpha'+n)/2}}$$

where $\alpha' = \operatorname{Re}\ \alpha$, $z = (1-\tau^2)/\varepsilon$, $\delta > 0$. If $z \leq 1/2\varepsilon$, i.e. $\tau^2 \geq 1/2$, then

$$|J^{\alpha}_{\varepsilon,1}| \lesssim \varepsilon^{\delta}|\tau| \int_{1/2}^{1-\tau^2/2} (1-u)^{-(\alpha'+1)/2}(u-\varepsilon z)^{\alpha'/2-1} \frac{du}{u^{\delta+(\alpha'+n)/2}}$$
$$\lesssim \varepsilon^{\delta}|\tau|^{-\alpha'} \int_{1/2}^{1-\tau^2/2} (u-\varepsilon z)^{\alpha'/2-1} du \lesssim \varepsilon^{\delta}.$$

If $z > 1/2\varepsilon$, i.e. $\tau^2 < 1/2$, then similarly we get

$$|J^{\alpha}_{\varepsilon,1}| \lesssim \varepsilon^{\delta}|\tau| \int_{1-\tau^2}^{1-\tau^2/2} (1-u)^{-(\alpha'+1)/2}(u-(1-\tau^2))^{\alpha'/2-1}du = \text{const } \varepsilon^{\delta}.$$

In order to construct a.c.$J^{\alpha}_{\varepsilon,2}$ and to estimate it, we use integration by parts. A simple calculation yields $J^{\alpha}_{\varepsilon,2} = \sum_{k=0}^{m-1}\sum_{p+q+r=k} a_{k,p,q,r} + \sum_{p+q+r=m} b_{p,q,r,m}$,

$$\begin{aligned} a_{k,p,q,r} \simeq & |\tau|^{2(k-r)}\varepsilon^{-p}(1-\tau^2/2)^{-(\alpha+n)/2-q}(I_+^{1+\beta-p}\mu)\Big(\frac{1-\tau^2/2}{\varepsilon}\Big), \\ b_{m,p,q,r} \simeq & \varepsilon^{-m+r+1-(\alpha+n)/2}|\tau| \\ & \times \int_{(1-\tau^2/2)/\varepsilon}^{1/\varepsilon} (1-\varepsilon s)^{m-(\alpha+1)/2}(I_+^{1+\beta-p}\mu)(s)\frac{(\tau^2-1+\varepsilon s)^{\alpha/2-1-r}ds}{s^{(\alpha+n)/2+q}}, \end{aligned}$$

$\alpha \neq 1,3,5,\dots$; $\beta = (n+\alpha-1)/2$, $m \in \mathbb{N}$. By Lemma 1.3, $(I_+^{1+\beta-p}\mu)(s) = O(s^{-p-\delta})$, $s > 1$, for some $\delta > 0$, and therefore $|a_{k,p,q,r}| \lesssim \varepsilon^{\delta}$. Similarly for $\alpha' = \text{Re } \alpha < 2m+1$ we get (use the inequalities $\tau^2/2 \leq \tau^2 - 1 + \varepsilon s \leq \tau^2$ and $1-\tau^2/2 > 1/2$)

$$|b_{m,p,q,r}| \lesssim \varepsilon^{\delta}|\tau|^{\alpha'-2m-1}\int_{1-\tau^2/2}^{1}\frac{(1-t)^{m-(\alpha'+1)/2}dt}{t^{(\alpha'+n)/2+m-r+\delta}} \lesssim \varepsilon^{\delta}.$$

The constant multiples, which are hidden in these estimates and depend on α, are uniformly bounded for α belonging to an arbitrary compact domain in the strip $0 < \text{Re } \alpha < 2m+1$. This provides the validity of (4.3) in this strip with the required estimate (4.6).

The statement (b) was, in fact, proved in Theorem 3.6. ∎

Proof of Theorem A: By Lemma 4.2, the equality (4.3) can be extended to $f \in L^p$, $1 \leq p < \infty$, and $f \in C$. It remains to apply the standard approximation procedure, which is based on (4.4)-(4.6) and the properties of the maximal function f^*. ∎

For $\text{Re}\,\alpha > 0$ the operator T^{α} is well-defined on the space $\mathcal{M}$ of finite Borel measures on Σ_n. Denote $(\nu,\omega) = \int_{\Sigma_n}\omega(x)d\nu(x)$, $\nu \in \mathcal{M}$. In the following we deal with "even" measures $\nu \in \mathcal{M}$ only, such that $(\nu,\omega) = (\nu(x),\omega(-x))$, $\omega \in C = C(\Sigma_n)$. For the set of all such measures we keep the same notation $\mathcal{M}$.

THEOREM 4.4: *Let* $\text{Re}\,\alpha > 0$, $\varphi = T^{\alpha}\nu$, $\nu \in \mathcal{M}$. *If* μ *satisfies* (0.11), (0.12), *and* $c_{\alpha,\mu}$ *is defined by* (0.14), *then*

$$c_{\alpha,\mu}(\nu,\omega) = \lim_{\varepsilon\to 0}\Big(\int_{\varepsilon}^{\infty}(W_{\mu}\varphi)(x,t)\frac{dt}{t^{1+(n+\alpha-1)/2}},\omega\Big), \quad \forall\omega \in C. \tag{4.14}$$

Proof: Owing to the convolution structure of all operators involved in our consideration, by Lemma 4.2 we have $(\mathcal{T}_\varepsilon^\alpha T^\alpha \nu, \omega) = (\nu, \mathcal{T}_\varepsilon^\alpha T^\alpha \omega) = (\nu, A_{\varepsilon,1}^\alpha \omega) + (\nu, A_{\varepsilon,2}^\alpha \omega) \to c_{\alpha,\mu}(\nu, \omega)$ as $\varepsilon \to 0$. This implies (4.14). ∎

THEOREM 4.5 (characterization of the ranges $T^\alpha(L^p)$, $T^\alpha(\mathcal{M})$): *Assume that* $\mathrm{Re}\ \alpha > 0$, $1 \le p \le \infty$, *and* μ *satisfies* (0.11), (0.12) *with* $c_{\alpha,\mu} \ne 0$, *(see* (0.14)*).*

(i) *For* $\varphi \in L^p$ *the following statements are equivalent:* (a) $\varphi \in T^\alpha(L^p)$; (b) *the integrals* $\mathcal{T}_\varepsilon^\alpha \varphi$ *(see* (3.14)*) converge in the* L^p*-norm.*

If $1 < p < \infty$, *then* (a) *and* (b) *are equivalent to:* (c) $\sup_{0<\varepsilon<1/2} \|\mathcal{T}_\varepsilon^\alpha \varphi\|_p < \infty$.

(ii) *For* $\varphi \in L^1$ *the following statements are equivalent:* (a′) $\varphi \in T^\alpha(\mathcal{M})$; (b′) *the sequence* $\int_{\Sigma_n} (\mathcal{T}_\varepsilon^\alpha \varphi)(x)\omega(x)dx$ *converges as* $\varepsilon \to 0$ *for arbitrary* $\omega \in C$.

If $\varphi = T^\alpha \nu$ *where* $\nu (\in \mathcal{M})$ *is nonnegative, then:* (c′) $\sup_{0<\varepsilon<1/2} \|\mathcal{T}_\varepsilon^\alpha \varphi\|_1 < \infty$.

If for $\varphi \in L^1$ *the relation* (c′) *holds, then* $\varphi \in T^\alpha(\mathcal{M})$.

Proof: (i) The implication (a) $\Rightarrow$ (b) follows from Theorem A. The validity of "(a)$\Rightarrow$(c)" is a consequence of Lemma 4.2. In order to prove "(b)$\Rightarrow$(a)" we denote

$$f = c_{\alpha,\mu}^{-1} \overset{(L^p)}{\lim_{\varepsilon\to 0}} \mathcal{T}_\varepsilon^\alpha \varphi.$$

Clearly, f is even. Then

$$T^\alpha f = c_{\alpha,\mu}^{-1} \overset{(L^p)}{\lim_{\varepsilon\to 0}} T^\alpha \mathcal{T}_\varepsilon^\alpha \varphi = c_{\alpha,\mu}^{-1} \overset{(L^p)}{\lim_{\varepsilon\to 0}} \mathcal{T}_\varepsilon^\alpha T^\alpha \varphi = \varphi$$

(here the L^p-boundedness of T^α and Theorem 4.3 have been used). Let us prove "(c)$\Rightarrow$(a)". Since the ball in L^p is compact in the weak* topology, there exist a sequence $\varepsilon_k \to 0$ and a function $f_0 \in L^p$ such that $\lim_{\varepsilon_k\to 0}(\mathcal{T}_{\varepsilon_k}^\alpha \varphi, \psi) = (f_0, \psi)$ for each $\psi \in L^{p'}$. Since the functions $\mathcal{T}_{\varepsilon_k}^\alpha \varphi$ are even, then f_0 is also even. Put $f = c_{\alpha,\mu}^{-1} f_0$. Then

$$(T^\alpha f, \psi) = (f, T^\alpha \psi) = \lim_{\varepsilon_k\to 0} c_{\alpha,\mu}^{-1}(\mathcal{T}_{\varepsilon_k}^\alpha \varphi, T^\alpha \psi) = \lim_{\varepsilon_k\to 0} c_{\alpha,\mu}^{-1}(\mathcal{T}_{\varepsilon_k}^\alpha T^\alpha \varphi, \psi) = (\varphi, \psi),$$

i.e. $\varphi = T^\alpha f$.

(ii) The implication (a′)$\Rightarrow$(b′) follows from Theorem 4.4. In order to prove "(a′) $\Rightarrow$ (c′)" we use Lemma 3.5 according to which $|(\mathcal{T}_\varepsilon^\alpha T^\alpha \nu, f)| = |(\nu, \mathcal{T}_\varepsilon^\alpha T^\alpha f)| \le \mathrm{const}\|f\|_\infty \|\nu\|_1\ \forall f \in C^\infty$. Since ν is nonnegative, for $f \equiv 1$ this relation reads $\|\mathcal{T}_\varepsilon^\alpha T^\alpha \nu\|_1 \le c\|\nu\|$ where $c \equiv$ const is independent of ε. Let us prove "(b′)=(a′)". Since the space of finite Borel measures on Σ_n is weakly complete, then there is a finite Borel measure ν such that $\lim_{\varepsilon\to 0}(\mathcal{T}_\varepsilon^\alpha \varphi, \omega) = (\nu, \omega)$. Obviously, ν is even.

Furthermore, for arbitrary infinitely differentiable function ψ, by Theorem A we have

$$(T^\alpha \nu, \psi) = (\nu, T^\alpha \psi) = \lim_{\varepsilon\to 0}(\mathcal{T}_\varepsilon^\alpha \varphi, T^\alpha \psi) = \lim_{\varepsilon\to 0}(\mathcal{T}_\varepsilon^\alpha T^\alpha \varphi, \psi) = c_{\alpha,\mu}(\varphi, \psi).$$

This implies $\varphi \overset{a.e.}{=} c_{\alpha,\mu}^{-1} T^\alpha \nu$. The proof of the implication "(c′)⇒(a′)" is similar to that of "(c)⇒(a)". ∎

5. L^p-theory (the case $\mathrm{Re}\,\alpha \le 0$)

By Corollary 2.6, the multiplier operator T^α is bounded in L^p for

$$(1-n)/2 + |1/p - 1/2|(n-1) \le \mathrm{Re}\,\alpha \le 0, \quad 1 < p < \infty.$$

Below we obtain a direct representation of $T^\alpha f$, $f \in L^p$, and solve the equation $T^\alpha f = \varphi$ explicitly. Let us start with the inversion problem. Our consideration is based on analytic continuation of the equality

$$\mathcal{T}_\varepsilon^\alpha T^\alpha f = \int_{\Sigma_n} \Lambda_\varepsilon^\alpha(xy) f(y) dy, \quad 0 < \mathrm{Re}\ \alpha < 1, \tag{5.1}$$

$$\Lambda_\varepsilon^\alpha(\tau) = \frac{|\tau|\ \Gamma((n+\alpha)/2)}{2\pi^{n/2}\Gamma(\alpha/2)} \times \int_{(1-\tau^2)/\varepsilon}^{1/\varepsilon} \lambda_\alpha(s)(1-\varepsilon s)^{-(\alpha+1)/2}(\varepsilon s)^{1-(\alpha+n)/2}(\tau^2 - 1 + \varepsilon s)^{\alpha/2-1} ds, \tag{5.2}$$

to the domain Re $\alpha \le 0$ (cf. (4.3), (4.7)–(4.9)), which is possible for $\lambda_\alpha(s)$ sufficiently smooth. If α and n are such that $\lambda_\alpha(s)$ is not smooth enough, we could achieve the required smoothness of λ_α, by putting $\mu = I_+^\theta \mu_0$ for some measure μ_0 and some $\theta > 0$ depending on α and n (see Remark 3.4). In fact the situation is more complicated because we want to extend (5.1) analytically so that the relevant L^p-theory will be applicable.

LEMMA 5.1: *Let $1 - n \le \mathrm{Re}\,\alpha \le 0$, $\ell = [-\mathrm{Re}\,\alpha/2] + 1$. Fix $\theta \ge 0$ so that*

$$\theta \ge \ell - \mathrm{Re}\ \beta = [-\mathrm{Re}\ \alpha/2] - \mathrm{Re}\ \alpha/2 + (3-n)/2, \tag{5.3}$$

and put $\mu = I_+^\theta \mu_0$ where μ_0 satisfies the following conditions:
(a)

$$\int_0^\infty s^j d\mu_0(s) = 0 \quad \forall j = 0, 1, \dots, m; \tag{5.4}$$

$$m = \begin{cases} [\mathrm{Re}\,\beta + \theta] & \text{if } \{\mathrm{Re}\,\beta + \theta\} < 1/2, \\ [\mathrm{Re}\ \beta + \theta] + 1 & \text{if } \{\mathrm{Re}\,\beta + \theta\} \ge 1/2; \end{cases}$$

(b)

$$\int_1^\infty s^\gamma d|\mu_0|(s) < \infty, \quad \gamma > \begin{cases} \text{Re } \beta + \theta + 1/2 & \text{if } \{\text{Re } \beta + \theta\} < 1/2, \\ \text{Re } \beta + \theta + 1 & \text{if } \{\text{Re } \beta + \theta\} \geq 1/2. \end{cases} \tag{5.5}$$

There exist spherical convolution operators $B^\alpha_{\varepsilon,1}, B^\alpha_{\varepsilon,2}$ *such that*

(i) *if* $f \in C^\infty$, $0 < \varepsilon < 1/2$, *then the analytic continuation of* (5.1) *is represented by*

$$\mathcal{T}^\alpha_\varepsilon T^\alpha f = B^\alpha_{\varepsilon,1} f + B^\alpha_{\varepsilon,2} f; \tag{5.6}$$

(ii) *if* $f \in L^p$, $1 \leq p \leq \infty$, *then*

$$\sup_{0<\varepsilon<1/2} |B^\alpha_{\varepsilon,1} f| \lesssim f^*, \qquad \sup_{0<\varepsilon<1/2} \|B^\alpha_{\varepsilon,1} f\|_p \lesssim \|f\|_p, \tag{5.7}$$

$$\sup_x |(B^\alpha_{\varepsilon,2} f)(x)| \lesssim \varepsilon^\delta \|f\|_p \quad \text{for some } \delta > 0. \tag{5.8}$$

Proof: We write (5.2) in the form (put $\tau^2 - 1 + \varepsilon s = \tau^2 \eta$, $\tau \neq 0$)

$$\begin{aligned} \Lambda^\alpha_\varepsilon(\tau) &= \frac{1}{\Gamma(\alpha/2)} \int_0^1 (I^{1+\beta}_+ \mu)\Big(\frac{1-\tau^2(1-\eta)}{\varepsilon}\Big)\Big(\frac{1-\tau^2(1-\eta)}{\varepsilon}\Big)^{-(\alpha+n)/2} \\ &\qquad \times \frac{\eta^{\alpha/2-1}(1-\eta)^{-(\alpha+1)/2}}{\varepsilon^{(\alpha+n)/2}} d\eta \\ &= \Big(\int_0^{1/2} + \int_{1/2}^1\Big)(\dots) = \overset{\circ}{K}{}^\alpha_{\varepsilon,1}(\tau) = \overset{\circ}{K}{}^\alpha_{\varepsilon,2}(\tau). \end{aligned} \tag{5.9}$$

By letting $(\overset{\circ}{B}{}^\alpha_{\varepsilon,i} f)(x) = \int_{\Sigma_n} \overset{\circ}{K}{}^\alpha_{\varepsilon,i}(xy) f(y) dy$, $i = 1, 2$, we get

$$\mathcal{T}^\alpha_\varepsilon T^\alpha f = \overset{\circ}{B}{}^\alpha_{\varepsilon,1} f + \overset{\circ}{B}{}^\alpha_{\varepsilon,2} f, \qquad 0 < \text{Re } \alpha < 1, \quad f \in C^\infty. \tag{5.10}$$

Our goal is to extend (5.10) to $\text{Re}\,\alpha \leq 0$ and to estimate the resulting expression.

The integrand in $\overset{\circ}{K}{}^\alpha_{\varepsilon,2}(\tau)$ has no singularity for $-n < \text{Re}\,\alpha < 1$ and represents the analytic function of α in this strip. Since $I^{1+\beta}_+ \mu = I^{1+\beta+\theta}_+ \mu_0$, then, by (5.4) and (5.5), due to (1.13) we have $(I^{1+\beta}_+ \mu)(s) = O(s^{-\delta_0})$ for some $\delta_0 > 0$. This yields (we omit simple calculations)

$$|\overset{\circ}{K}{}^\alpha_{\varepsilon,2}(\tau)| \lesssim \varepsilon^{\delta_0}, \quad 0 < \varepsilon < 1/2, \tag{5.11}$$

uniformly in α belonging to an arbitrarily fixed compact domain in the strip $-n < \text{Re}\,\alpha < 1$. In order to handle the first term in (5.9) we use integration by parts, which gives (up to constant multiples having a nice behaviour in the α-variable)

$$\overset{\circ}{K}{}^{\alpha}_{\varepsilon,1} = \sum_{k=0}^{\ell-1} \sum_{p+q+r=k} a^{\alpha,\varepsilon}_{k,p,q,r} + \sum_{p+q+r=\ell} b^{\alpha,\varepsilon}_{\ell,p,q,r}, \quad 0 < \text{Re } \alpha < 1, \tag{5.12}$$

$$a^{\alpha,\varepsilon}_{k,p,q,r}(\tau) = \varepsilon^{-p}(\tau^2)^{p+q}(1-\tau^2/2)^{-(\alpha+n)/2-q}\, g^{\alpha}_p((1-\tau^2/2)/\varepsilon), \tag{5.13}$$

$$b^{\alpha,\varepsilon}_{\ell,p,q,r}(\tau) = \frac{(\tau^2)^{p+q}}{\varepsilon^{p+q+(\alpha+n)/2}} \int_0^{1/2} g^{\alpha}_p\Big(\frac{1-\tau^2(1-\eta)}{\varepsilon}\Big)\Big(\frac{1-\tau^2(1-\eta)}{\varepsilon}\Big)^{-(\alpha+n)/2-q} \times \eta^{\alpha/2+\ell-1}(1-\eta)^{-(\alpha+1)/2-r}d\eta. \tag{5.14}$$

Here $g^{\alpha}_p(s) = (I^{1+\beta-p}_{+}\mu)(s) = (I^{1+\beta+\theta-p}_{+}\mu_0)(s)$, ℓ and θ are the same as in (5.3). We observe that by (5.3), $\text{Re}(\beta+\theta-p) \geq 0$ for all $p \leq \ell$. Owing to (5.4) and (5.5), by Lemma 1.3 we have

$$|g^{\alpha}_p(s)| = \begin{cases} O(s^{\beta'+\theta-p}) & \text{if } s \leq 1, \\ O(s^{-p-\delta}) & \text{if } s > 1, \end{cases} \quad \beta' = \text{Re}\,\beta, \tag{5.15}$$

$$\delta = \left\{ \begin{array}{ll} \min(\gamma-\beta'-\theta,\ 1-\{\beta'+\theta\}) & \text{if } \{\beta'+\theta\} < 1/2, \\ 1+\min(\gamma-\beta'-\theta-1,\ 1-\{\beta'+\theta\}) & \text{if } \{\beta'+\theta\} \geq 1/2 \end{array} \right\} > 1/2. \tag{5.16}$$

By (5.15), $|a^{\alpha,\varepsilon}_{k,p,q,r}(\tau)| \lesssim \varepsilon^{\delta}$ uniformly in $\tau \in [-1,1]$ and $1-n \leq \text{Re}\,\alpha < 1$. The expressions $\overset{\circ}{K}{}^{\alpha}_{\varepsilon,2}(\tau)$ and $a^{\alpha,\varepsilon}_{k,p,q,r}(\tau)$ constitute the second term in (5.6) for which (5.8) is valid.

Consider $b^{\alpha,\varepsilon}_{\ell,p,q,r}$. For $|\tau| < 1$ and ℓ fixed, this expression represents the analytic function of α at least for $\max(-n,-2\ell) < \text{Re}\,\alpha < 1$. In order to estimate $b^{\alpha,\varepsilon}_{\ell,p,q,r}$ we denote $\alpha' = \text{Re}\,\alpha$, $\Delta = 1-\tau^2$, $z = \Delta/\varepsilon$, and use the same scheme as in the proof of Lemma 4.2. If $z < 1$, then $1/2 > \varepsilon > 1-\tau^2$, $\tau^2 > 1/2$, and we proceed as follows:

$$b^{\alpha,\varepsilon}_{\ell,p,q,r} = \frac{(\tau^2)^{p+q}}{\varepsilon^{p+q+(\alpha+n)/2}}\Big(\int_0^{(\tau^2-(1-\varepsilon))/\tau^2} + \int_{(\tau^2-(1-\varepsilon))/\tau^2}^{1/2}\Big)(\dots) = J^{\alpha}_1 + J^{\alpha}_2. \tag{5.18}$$

By (5.15),

$$|J^{\alpha}_1| \lesssim \varepsilon^{-p-q-(\alpha'+n)/2} \int_0^{(\varepsilon-\Delta)/\tau^2} \eta^{\alpha'/2+\ell-1}\Big(\frac{\Delta+\tau^2\eta}{\varepsilon}\Big)^{-(\alpha'+n)/2-q+\beta'+\theta-p} d\eta$$

(change the variable: $\Delta + \tau^2\eta = \Delta/s$)

$$\lesssim \varepsilon^{-\beta'-\theta}\Delta^{\beta'+\theta+r-n/2} \int_{\Delta/\varepsilon}^{1} \frac{(1-s)^{\alpha'/2+\ell-1}ds}{s^{\beta'+\theta+r+1-n/2}}, \quad \ell = r+p+q. \tag{5.19}$$

If $\Delta/\varepsilon \geq 1/2$, then *for all* $\delta_0 > 0$,

$$(5.20) \qquad |J_1^\alpha| \lesssim \Delta^{r-n/2} = \frac{\Delta^{1-n/2}}{\varepsilon}\Big(\frac{\Delta}{\varepsilon}\Big)^{\delta_0-1}\Big(\frac{\varepsilon}{\Delta}\Big)^{\delta_0}\Delta^r \lesssim \frac{\Delta^{1-n/2}}{\varepsilon}\Big(\frac{\Delta}{\varepsilon}\Big)^{\delta_0-1}.$$

If $\Delta/\varepsilon < 1/2$, a simple estimation of the integral in (5.19) gives the same result *for some* $\delta_0 > 0$. Similarly by (5.15) we obtain

$$(5.21) \qquad |J_2^\alpha| \lesssim \varepsilon^\delta \Delta^{r-n/2-\delta}\int_{2\Delta/(1+\Delta)}^{\Delta/\varepsilon} s^{n/2-r+\delta-1}(1-s)^{\alpha'/2+\ell-1}ds.$$

If $\Delta/\varepsilon < 1/2$, then for $r < n/2+\delta$ (since $\delta > 1/2$, this inequality holds for all $r \leq \ell$),

$$|J_2^\alpha| \lesssim \varepsilon^\delta\Delta^{r-n/2-\delta}\Big(\frac{\Delta}{\varepsilon}\Big)^{n/2-r+\delta} \leq \frac{\Delta^{1-n/2}}{\varepsilon}\Big(\frac{\Delta}{\varepsilon}\Big)^{n/2-1}.$$

If $1/2 \leq \Delta/\varepsilon$ (< 1), then $|J_2^\alpha| \lesssim \Delta^{r-n/2}$ and we proceed as in (5.20). Let $z = \Delta/\varepsilon \geq 1$. By (5.15),

$$\begin{aligned} |b_{\ell,p,q,r}^{\alpha,\varepsilon}| &\lesssim \frac{(\tau^2)^{p+q}}{\varepsilon^{p+q+(\alpha'+n)/2}}\int_0^{1/2}\eta^{\alpha'/2+\ell-1}\Big(\frac{\Delta+\tau^2\eta}{\varepsilon}\Big)^{-(\alpha'+n)/2-q-p-\delta}d\eta \\ &= \frac{(\tau^2)^{-r-\alpha'/2}\varepsilon^\delta}{\Delta^{n/2+\delta-r}}r(\tau), \end{aligned}$$

where

$$r(\tau) = \int_0^{\tau^2/2\Delta} s^{\alpha'/2+\ell-1}(1+s)^{-(\alpha'+n)/2+r-\ell-\delta}\,ds.$$

If $\Delta \geq 1/2$, then $r(\tau) \leq \int_0^{\tau^2}(\dots)$, and we get

$$|b_{\ell,p,q,r}^{\alpha,\varepsilon}| \lesssim \varepsilon^\delta \leq \varepsilon^{-1}\Delta^{1-n/2}(\Delta/\varepsilon)^{-1-\delta}.$$

If $\Delta < 1/2$, then $r(\tau) < r(\infty) < \infty$, and therefore

$$|b_{\ell,p,q,r}^{\alpha,\varepsilon}| \lesssim \varepsilon^\delta\Delta^{r-n/2-\delta} \leq \varepsilon^{-1}\Delta^{1-n/2}(\Delta/\varepsilon)^{-1-\delta}.$$

The second sum in (5.12) gives the first term $B_{\varepsilon,1}^\alpha f$ in (5.6). Moreover, $(B_{\varepsilon,1}^\alpha f)(x) = \int_{\Sigma_n} K_{\alpha,1}^\alpha(xy)f(y)dy$ where $K_{\alpha,1}^\alpha(\tau)$ is a kernel similar to that in (4.10). This implies (5.7) and (5.6). ∎

Remark 5.2: An examination of the estimates of J_2^α and $b_{\ell,p,q,r}^{\alpha,\varepsilon}$ ($z \geq 1$) shows that, for Re $\alpha \geq (1-n)/2$, it suffices to assume $m = [\operatorname{Re}\beta+\theta]$, $\gamma > \operatorname{Re}\beta+\theta$ in all situations.

THEOREMS 5.3: *Let* $(1-n)/2+|1/p-1/2|(n-1)\le \mathrm{Re}\,\alpha\le 0$, $1<p<\infty$. *Assume that* μ *is the wavelet measure defined in Lemma* 5.1 (*see also Remark* 5.2).

(i) *If* $\varphi = T^\alpha f$, $f\in L^p$, *where* T^α *is the "L^p-extension" of the operator* (2.4), *then the inversion formula* (0.13) *is valid.*

(ii) *If* $c_{\alpha,\mu}\ne 0$ (*see* (0.14)), *then for* $\varphi\in L^p$ *the following statements are equivalent:* (a) $\varphi\in T^\alpha(L^p)$; (b) *the integrals* $\mathcal{T}_\varepsilon^\alpha\varphi$ *converge in the* L^p*-norm;* (c) $\sup_{0<\varepsilon<1/2}\|\mathcal{T}_\varepsilon^\alpha\varphi\|_p<\infty$.

The proof is similar to that of Theorems A and 4.5 (use Lemma 5.1, Theorem 3.6).

Proof of Theorem B: Given $f\in L^p$, let $\{f_j\}$ be a sequence of even C^∞-functions approximating f in the L^p-norm. Denote $\tilde\alpha = 1-n-\alpha$ so that Re $\tilde\alpha\in[1-n,\ (1-n)/2]$. By Lemma 5.1 (with α replaced by $\tilde\alpha$) and the equality $T^{\tilde\alpha}T^\alpha f_j = f_j$ we get

$$\int_\varepsilon^\infty (Wf)(x,t)\frac{dt}{t^{1-\alpha/2}} \overset{(3.8)}{=} \overset{(L^p)}{\lim_{j\to\infty}} \int_\varepsilon^\infty (Wf_j)(x,t)\frac{dt}{t^{1-\alpha/2}}$$

$$= \overset{(L^p)}{\lim_{j\to\infty}} \mathcal{T}_\varepsilon^{\tilde\alpha}T^{\tilde\alpha}T^\alpha f_j \overset{(5.6)}{=} \overset{(L^p)}{\lim_{j\to\infty}} [B^{\tilde\alpha}_{\varepsilon,1}T^\alpha f_j + B^{\tilde\alpha}_{\varepsilon,2}T^\alpha f_j] = B^{\tilde\alpha}_{\varepsilon,1}T^\alpha f + B^{\tilde\alpha}_{\varepsilon,2}T^\alpha f.$$

Owing to (5.7) and (5.8) the required result then follows in a standard way. ∎

References

[1] W. Blaschke, *Kreis und Kugel*, Chelsea, New York, 1949.

[2] A. Erdélyi (ed.), *Higher Transcendental Functions*, Vols. I, II, McGraw-Hill, New York, 1953.

[3] R. J. Gardner, *Geometric Tomography*, Cambridge University Press, 1995.

[4] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals Series and Products*, Academic Press, New York, 1980.

[5] S. Helgason, *Geometric Analysis on Symmetric Spaces*, American Mathematical Society, Providence, RI, 1994.

[6] A. Koldobsky, *Inverse formula for the Blaschke–Levy representation*, Houston Journal of Mathematics **23** (1997), 95–107.

[7] V. S. Kryuchkov, *Differential properties of the symbol of a Calderón–Zygmund singular integral operator*, Proceedings of the Steklov Institute of Mathematics **170** (1987), Issue 1, 169–183.

[8] P. Levy, *Théory de l'addition de variable aléatoires*, Gauthier-Villars, Paris, 1937.

[9] S. Meda and R. Pini, *Spherical convolutions with kernels having singularities on an equator,* Unione Matematica Italiana Bollettino. (7) **5-B** (1991), 275–290.

[10] A. P. Prudnikov, Y. A. Brychkov and O. I. Marichev, *Integrals and Series: Special Functions,* Gordon and Breach, New York–London, 1986.

[11] B. Rubin, *Fractional Integrals and Potentials,* Addison Wesley Longman, Essex, U.K., 1996.

[12] B. Rubin, *Spherical Radon transforms and related wavelet transforms,* Applied and Computational Harmonic Analysis **5** (1998), 202–215.

[13] B. Rubin, *Inversion of fractional integrals related to the spherical Radon transform,* Journal of Functional Analysis **157** (1998), 470–487.

[14] B. Rubin, *Fractional calculus and wavelet transforms in integral geometry,* Fractional Calculus and Applied Analysis **1** (1998), No. 2, 193–219.

[15] S. G. Samko, *Singular integrals over a sphere and the construction of the characteristic from the symbol,* Soviet Mathematics (Izvestiya VUZ) **27** (1983), No. 4, 35–52.

[16] V. I. Semyanistyi, *Some integral transformations and integral geometry in an elliptic space,* (Russian) Trudy Seminara po Vektornomu i Tenzornomu Analizu **12** (1963), 397–441.

[17] R. S. Strichartz, *Multipliers for spherical harmonic expansions,* Transactions of the American Mathematical Society **167** (1972), 115–124.